NONNEGATIVE POLYNOMIALS AND CIRCUIT POLYNOMIALS

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Abstract. Circuit polynomials are polynomials supported on circuits. The nonnegativity of circuit polynomials is easy to check. Representing polynomials as sums of nonnegative circuit polynomials is a certificate of the nonnegativity of polynomials. For a polynomial with a simplex Newton polytope satisfying certain conditions, it is nonnegative if and only if it is a sum of nonnegative circuit polynomials. In this paper, we generalize this conclusion to polynomials with general Newton polytopes. Moreover, we put the problem to decide if a polynomial can be written as a sum of nonnegative circuit polynomials down to the feasibility of a relative entropy program. Since relative entropy programs are convex, they can be checked very efficiently.

1. Introduction

A real polynomial $p \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ is called a nonnegative polynomial if its evaluation on every real point is nonnegative. All of nonnegative polynomials form a convex cone, denoted by PSD. Certifying the nonnegativity of a polynomial p is a central problem of real algebraic geometry and also has important applications to optimization problems. The classical method for this problem is writing p as a sum of squares of polynomials (SOS), and then p is obviously nonnegative. The key idea of this method is representing p as a sum of a certain class of nonnegative polynomials whose nonnegativity is easy to check.

Recently in [7], Iliman and Wolff introduced the concept of nonnegative circuit polynomials as a substitute of squares of polynomials to represent p. A polynomial f is called a *circuit polynomial* if it is of the form

$$f(\mathbf{x}) = \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}}, \tag{1.1}$$

where the Newton polytope $\Delta = \mathrm{New}(f)$ is a lattice simplex with the vertex set $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}$, $\boldsymbol{\beta}$ an interior point of Δ and $c_i > 0$ for $i = 1, \ldots, m$. For every circuit polynomial f, we associate it with the *circuit number* defined as $\Theta_f := \prod_{i=1}^{m}(c_i/\lambda_i)^{\lambda_i}$, where the λ_i's are uniquely given by the convex combination $\boldsymbol{\beta} = \sum_{i=1}^{m} \lambda_i \boldsymbol{\alpha}_i$ with $\lambda_i > 0$ and $\sum_{i=1}^{m} \lambda_i = 1$. The nonnegativity of circuit polynomials is easy to check. Actually the circuit polynomial f is nonnegative if and only if $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n$ for all i, and $-\Theta_f \leq d \leq \Theta$ if $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ or $d \geq -\Theta$ if $\boldsymbol{\beta} \in (2\mathbb{N})^n$.

If a polynomial p can be written as a sum of nonnegative circuit polynomials, then p is obviously nonnegative. All of polynomials which is a sum of nonnegative circuit

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polynomials also form a convex cone, denoted by SONC. Based on these SONC decompositions for nonnegativity certificates, new approaches were proposed for both unconstrained polynomial optimization problems and constrained polynomial optimization problems, which were proven to be significantly more efficient than the classic semidefinite programming method in many cases ([3, 4, 5, 8]).

Clearly, the inclusion SONC $\subseteq$ PSD holds. It is natural to ask which type of nonnegative polynomials has a SONC decomposition and how big the gap between PSD and SONC is. In [7], it was proven that if the Newton polytope $\mathrm{New}(p)$ is a simplex and there exists a point such that all terms of p except for those corresponding to vertices have the negative sign on this point, then $p \in$ PSD if and only if $p \in$ SONC. In this paper, we generalize this conclusion to polynomials with general Newton polytopes. We prove

Theorem 1.1. *Let $p = \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^{l} d_j \mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$, $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 1,\dots,m$, $\boldsymbol{\beta}_j \in \mathrm{New}(p)^{\circ} \cap \mathbb{N}^n, j = 1,\dots,l$. Assume that all of the $\boldsymbol{\beta}_j$'s lie in the same side of all hyperplanes through any n points of $\{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_m\}$ and there exists a point $\boldsymbol{v} = (v_j) \in (\mathbb{R}^*)^n$ such that $d_j \boldsymbol{v}^{\boldsymbol{\beta}_j} > 0$ for all j. Then $p \in$ PSD if and only if $p \in$ SONC.*

The conditions that all of the $\boldsymbol{\beta}_j$'s lie in the same side of all hyperplanes through any n points of $\{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_m\}$ and there exists a point $\boldsymbol{v} = (v_j) \in (\mathbb{R}^*)^n$ such that $d_j \boldsymbol{v}^{\boldsymbol{\beta}_j} > 0$ for all j in Theorem 1.1 are necessary. We will give counterexamples to illustrate this.

From the perspective of computation, we put the problem to decide $p \in$ SONC down to the feasibility of a relative entropy program. Since relative entropy programs are convex, they can be checked very efficiently.

The rest of this paper organized as follows. In Section 2, we introduce some notions and recall some results on circuit polynomials. In Section 3, we deal with the case of nonnegative polynomials with one negative term. In Section 4, we deal with the case of nonnegative polynomials with multiple negative terms. In Section 5, we compute SONC decompositions of nonnegative polynomials via relative entropy programs.

2. Preliminaries

2.1. **Nonnegative Polynomials.** Let $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1,\dots,x_n]$ be the ring of real n-variate polynomial, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$, and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. Let $\mathbb{R}_+$ be the set of positive real numbers and $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers. For a finite set $A \subset \mathbb{N}^n$, we denote by $\mathrm{cone}(A)$ the conic hull of A, by $\mathrm{conv}(A)$ the convex hull of A, and by $V(A)$ the vertices of the convex hull of A. Also we denote by $V(P)$ the vertex set of a polytope P. We consider polynomials $f \in \mathbb{R}[\mathbf{x}]$ supported on $A \subset \mathbb{N}^n$, i.e. f is of the form $f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in A} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ with $c_{\boldsymbol{\alpha}} \in \mathbb{R}, \mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The Newton polytope is defined as $\mathrm{New}(f) = \mathrm{conv}(\{\boldsymbol{\alpha} \in A : c_{\boldsymbol{\alpha}} \neq 0\})$. For a polytope P, we use P° to denote the interior of P.

A polynomial $f \in \mathbb{R}[\mathbf{x}]$ which is nonnegative over $\mathbb{R}^n$ is called a nonnegative polynomial. The class of nonnegative polynomials is denoted by PSD, which is a convex cone.

A nonnegative polynomial must satisfy the following necessary conditions.

Proposition 2.1 ([9])**.** *Let $A \subset \mathbb{N}^n$ be a finite set and $f = \sum_{\boldsymbol{\alpha} \in A} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{R}[\mathbf{x}]$ be supported on A. Then f is nonnegative only if the following hold:*

(1) $V(A) \subset (2\mathbb{N})^n$*;*
(2) If $\boldsymbol{\alpha} \in V(A)$*, then the corresponding coefficient* $c_{\boldsymbol{\alpha}}$ *is positive.*

For the remainder of this paper, we assume that these necessary conditions in Proposition 2.1 are satisfied. Furthermore, for simplicity, we assume that the monomial factor of f is 1, that is, if $f = \mathbf{x}^{\boldsymbol{\alpha}'}(\sum c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}})$ such that $\sum c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{R}[\mathbf{x}]$, then $\mathbf{x}^{\boldsymbol{\alpha}'} = 1$.

2.2. **Nonnegative Polynomials Supported on Circuits.**

Definition 2.2. *Let* $f \in \mathbb{R}[\mathbf{x}]$ *be supported on* $A \subset \mathbb{N}^n$ *such that all elements of* $V(A)$ *are in* $(2\mathbb{N})^n$*. Then* f *is called a* circuit polynomial *if it is of the form*

$$f(\mathbf{x}) = \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}], \tag{2.1}$$

with $m \le n+1$*,* $c_i > 0, i = 1, \ldots, m$ *such that the following conditions hold:*

(1) the set of points $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}$ *are affinely independent and equal* $V(A)$*;*
(2) The exponent $\boldsymbol{\beta}$ *can be written uniquely as*

$$\boldsymbol{\beta} = \sum_{i=1}^{m} \lambda_i \boldsymbol{\alpha}_i \text{ with } \lambda_i > 0 \text{ and } \sum_{i=1}^{m} \lambda_i = 1. \tag{2.2}$$

For every circuit polynomial f*, we define the corresponding* circuit number *as* $\Theta_f := \prod_{i=1}^{m}(c_i/\lambda_i)^{\lambda_i}$*.*

The nonnegative of a circuit polynomial f is decided by its circuit number alone.

Theorem 2.3. *(*[7, Theorem 3.8]*) Let* f *be a circuit polynomial and* Θ_f *its circuit number. Then* f *is nonnegative if and only if* $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ *and* $|d| \le \Theta_f$*, or* $\boldsymbol{\beta} \in (2\mathbb{N})^n$ *and* $d \le \Theta_f$*.*

Proposition 2.4. *(*[7, Proposition 3.4]*) Let* $f = \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - \Theta_f \mathbf{x}^{\boldsymbol{\beta}}$ *be a circuit polynomial,* Θ_f *the circuit number and* $\boldsymbol{\beta} = \sum_{i=1}^{m} \lambda_i \boldsymbol{\alpha}_i$ *with* $\lambda_i > 0$ *and* $\sum_{i=1}^{m} \lambda_i = 1$*. Then* f *has exactly one zero* $\mathbf{x}_*$ *in* $\mathbb{R}_+^n$ *which satisfies:*

$$\frac{c_1 \mathbf{x}_*^{\boldsymbol{\alpha}_1}}{\lambda_1} = \cdots = \frac{c_m \mathbf{x}_*^{\boldsymbol{\alpha}_m}}{\lambda_m} = \Theta_f \mathbf{x}_*^{\boldsymbol{\beta}}. \tag{2.3}$$

Moreover, if $\mathbf{x}$ *is any zero of* f*, then* $|\mathbf{x}| = \mathbf{x}_*$*, which means* $|x_i| = (x_*)_i$ *for* $i = 1, \ldots, n$*.*

In analogy with writing a polynomial as sums of squares, writing a polynomial as a sum of nonnegative circuit polynomials is a certificate of nonnegativity. We denote by SONC the class of polynomials which can be written as sums of nonnegative circuit polynomials.

Theorem 2.5. *(*[7, Corollary 7.5]*) Let* $f = \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^{l} d_j \mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$ *with* $c_i > 0, \boldsymbol{\alpha}_i \in (2\mathbb{N})^n, i = 1, \ldots, m$ *such that* $\Delta = \mathrm{conv}(\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\})$ *is a simplex and* $\boldsymbol{\beta}_j \in \Delta^\circ \cap \mathbb{N}^n$ *for* $j = 1, \ldots, l$*. If there exists a vector* $\boldsymbol{v} = (v_j) \in (\mathbb{R}^*)^n$ *such that* $d_j \boldsymbol{v}^{\boldsymbol{\beta}_j} > 0$ *for all* j*, then* f *is nonnegative if and only if* $f \in$ SONC*.*

3. Nonnegative Polynomials with One Negative Term

In this section, we consider nonnegative polynomials with only one possible negative term. For $m \in \mathbb{N}$, let $[m] := \{1, \ldots, m\}$.

Let $f_d = \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$, $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 1, \ldots, m$, $\boldsymbol{\beta} \in \mathrm{New}(f_d)^{\circ} \cap \mathbb{N}^n$. Without loss of generality, assume $\dim(\mathrm{New}(f_d)) = n$ and $m > n+1$. It is easy to see that the set $\{d \in \mathbb{R} \mid f_d \text{ is nonnegative}\}$ is nonempty and has upper bounds. So the supremum exists. Let

$$d^* \triangleq \sup\{d \in \mathbb{R} \mid f_d \text{ is nonnegative}\}. \tag{3.1}$$

Clearly f_{d^*} is nonnegative and has a zero.

Theorem 3.1. ([6, Theorem 1.5]) *Consider the following system of polynomial equations*

$$\sum_{i=1}^{m} c_i \boldsymbol{\alpha}_i \mathbf{x}^{\boldsymbol{\alpha}_i} = \mathbf{b}, \tag{3.2}$$

where $\boldsymbol{\alpha}_i \in \mathbb{R}^n, c_i > 0, 1 \le i \le m$. *Moreover, assume* $\dim \mathrm{conv}(\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}) = n$. *Then for any* $\mathbf{b} \in \mathrm{cone}(\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\})^{\circ}$, *(3.2) has exactly one zero in* $\mathbb{R}_+^n$.

Lemma 3.2. *Assume* $\dim \mathrm{conv}(\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}) = n$. *The following system of polynomial equations on variables* $(\mathbf{x}, d)$

$$\begin{cases} \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} = 0 \\ \sum_{i=1}^{m} c_i \boldsymbol{\alpha}_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\boldsymbol{\beta}\mathbf{x}^{\boldsymbol{\beta}} = \mathbf{0} \end{cases}, \tag{3.3}$$

where $\boldsymbol{\alpha}_i \in \mathbb{R}^n, c_i > 0, 1 \le i \le m$, $\boldsymbol{\beta} \in \mathrm{conv}(\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\})^{\circ}$, *has exactly one zero in* $\mathbb{R}_+^{n+1}$.

Proof. Eliminate d from (3.3) and we obtain

$$\sum_{i=1}^{m} c_i (\boldsymbol{\alpha}_i - \boldsymbol{\beta}) \mathbf{x}^{\boldsymbol{\alpha}_i} = \mathbf{0}. \tag{3.4}$$

Divide (3.4) by $\mathbf{x}^{\boldsymbol{\beta}}$, and we have

$$\sum_{i=1}^{m} c_i (\boldsymbol{\alpha}_i - \boldsymbol{\beta}) \mathbf{x}^{\boldsymbol{\alpha}_i - \boldsymbol{\beta}} = \mathbf{0}. \tag{3.5}$$

Since $\boldsymbol{\beta} \in \mathrm{conv}(\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\})^{\circ}$, we have $\mathbf{0} \in \mathrm{cone}(\{\boldsymbol{\alpha}_1 - \boldsymbol{\beta}, \ldots, \boldsymbol{\alpha}_m - \boldsymbol{\beta}\})^{\circ}$. Thus by Theorem 3.1, (3.5) and hence (3.4) have exactly one solution in $\mathbb{R}_+^n$, say $\mathbf{x}_*$. Substitute $\mathbf{x}_*$ into the first equation of (3.3), and we obtain $d = \sum_{i=1}^{m} c_i \mathbf{x}_*^{\boldsymbol{\alpha}_i - \boldsymbol{\beta}} > 0$. □

Theorem 3.3. *Let* $f_d = \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$, $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 1, \ldots, m$, $\boldsymbol{\beta} \in \mathrm{New}(f_d)^{\circ} \cap \mathbb{N}^n$, $\dim(\mathrm{New}(f_d)) = n$, *and* d^* *be defined as (3.1). Then* f_d *is nonnegative if and only if* $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ *and* $|d| \le d^*$, *or* $\boldsymbol{\beta} \in (2\mathbb{N})^n$ *and* $d \le d^*$. *Moreover,* f_{d^*} *has exactly one zero in* $\mathbb{R}_+^n$.

Proof. First, if $\boldsymbol{\beta} \in (2\mathbb{N})^n$ and $d \le 0$, then obviously f_d is nonnegative since it is a sum of monomial squares. If $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ and $d \le 0$, then f_d is nonnegative if and only if f_{-d} is nonnegative. So without loss of generality, we can always assume $d > 0$. Since the only possible negative term in f_d is $-d\mathbf{x}^{\boldsymbol{\beta}}$, f_d is nonnegative over $\mathbb{R}^n$ if and only if f_d is nonnegative over $\mathbb{R}_+^n$. Therefore, by the definition of d^*, f_d

is nonnegative for $d \le d^*$. The zeros of f_{d^*} are also the minimums of f_{d^*}. So they satisfy $f_{d^*}(\mathbf{x}) = \nabla(f_{d^*}(\mathbf{x})) = 0$ which is

$$\begin{cases} \sum_{i=1}^m c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} = 0 \\ \sum_{i=1}^m c_i \boldsymbol{\alpha}_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\boldsymbol{\beta}\mathbf{x}^{\boldsymbol{\beta}} = \mathbf{0} \end{cases}. \tag{3.6}$$

By Lemma 3.2, (3.6) has exactly one zero in $\mathbb{R}^n_+$, and so does f_{d^*}. □

We will need the following theorem from discrete geometry.

Theorem 3.4 (Helly, [2])**.** *Let $X_1, \dots, X_r$ be a finite collection of convex subsets of $\mathbb{R}^n$ with $r > n$. If the intersection of every $n+1$ of these sets is nonempty, then the whole collection has a nonempty intersection.*

Corollary 3.5. *Let $X_1, \dots, X_r$ be a finite collection of convex subsets of $\mathbb{R}^n$ with $r > n+1$. If the intersection of every $r-1$ of these sets is nonempty, then the whole collection has a nonempty intersection.*

Proof. Since $r > n+1$, the condition that the intersection of every $r-1$ of these sets is nonempty implies that the intersection of every $n+1$ of these sets is nonempty. So the corollary is immediate from Theorem 3.4. □

Lemma 3.6. *Let $A = (a_{ij}) \in \mathbb{R}^{m\times r}$, $\mathbf{b} = (b_j) \in \mathbb{R}^r$ and $\mathbf{z} = (z_1, \dots, z_r)^T$ a set of variables. For each j, let A_j be the submatrix by deleting all of the i-th rows with $a_{ij} \neq 0$ and the j-th column from A. Assume that $A\mathbf{z} = \mathbf{b}$ has a solution,* $\operatorname{rank}(A) > 1$ *and* $\operatorname{rank}(A_j) = \operatorname{rank}(A) - 1$ *for all j. Then $A\mathbf{z} = \mathbf{b}$ has a nonnegative solution if and only if $A_j\bar{\mathbf{z}}_j = \bar{\mathbf{b}}_j$ has a nonnegative solution for $j = 1, \dots, r$, where $\bar{\mathbf{z}}_j = \mathbf{z}\backslash z_j$, $\bar{\mathbf{b}}_j = \mathbf{b}\backslash b_j$.*

Proof. Let $t = \operatorname{rank}(A) > 1$. Then the system of linear equations $A\mathbf{z} = \mathbf{b}$ has $r-t$ free variables. Without loss of generality, let the $r-t$ free variables be $\{z_1, \dots, z_{r-t}\}$. We can figure out $\{z_{r-t+1}, \dots, z_r\}$ from $A\mathbf{z} = \mathbf{b}$ and assume $z_j = f_{j-r+t}(z_1, \dots, z_{r-t})$ for $j = r-t+1, \dots, r$. Then $A\mathbf{z} = \mathbf{b}$ has a nonnegative solution if and only if

$$\{(z_1, \dots, z_{r-t}) \mid z_1 \ge 0, \dots, z_{r-t} \ge 0, z_{r-t+1} = f_1 \ge 0, \dots, z_r = f_t \ge 0\} \tag{3.7}$$

is nonempty. Since both $z_j \ge 0, 1 \le j \le r-t$ and $f_j \ge 0, 1 \le j \le t$ define convex subsets of $\mathbb{R}^{r-t}$, then by Corollary 3.5, (3.7) is nonempty if and only if

$$\begin{aligned} \{(z_1, \dots, z_{r-t}) \mid & z_1 \ge 0, \dots, z_{j-1} \ge 0, z_{j+1} \ge 0, \dots, z_{r-t} \ge 0, \\ & z_{r-t+1} = f_1 \ge 0, \dots, z_r = f_t \ge 0\} \end{aligned} \tag{3.8}$$

is nonempty for $j = 1, \dots, r-t$ and

$$\begin{aligned} \{(z_1, \dots, z_{r-t}) \mid & z_1 \ge 0, \dots, z_{r-t} \ge 0, z_{r-t+1} = f_1 \ge 0, \dots, z_{j-1+r-t} = f_{j-1} \ge 0, \\ & z_{j+1+r-t} = f_{j+1} \ge 0, \dots, z_r = f_t \ge 0\} \end{aligned} \tag{3.9}$$

is nonempty for $j = 1, \dots, t$.

For $j \in [r-t]$, (3.8) is nonempty if and only if $A\mathbf{z} = \mathbf{b}$ has a solution with $\bar{\mathbf{z}}_j \in \mathbb{R}^{r-1}_{\ge 0}$ and $z_j \in \mathbb{R}$, which is equivalent to the condition that $A_j\bar{\mathbf{z}}_j = \bar{\mathbf{b}}_j$ has a nonnegative solution since $\operatorname{rank}(A_j) = \operatorname{rank}(A) - 1$. For $j \in [t]$, (3.9) is nonempty if and only if $\{f_1 \ge 0, \dots, f_{j-1} \ge 0, f_{j+1} \ge 0, \dots, f_t \ge 0\}$ has a nonnegative solution, which is also equivalent to the condition that $A_{j+r-t}\bar{\mathbf{z}}_{j+r-t} = \bar{\mathbf{b}}_{j+r-t}$ has a nonnegative solution since $\operatorname{rank}(A_{j+r-t}) = \operatorname{rank}(A) - 1$. Put all above together and we have that $A\mathbf{z}=\mathbf{b}$ has a nonnegative solution if and only if $A_j\bar{\mathbf{z}}_j=\bar{\mathbf{b}}_j$ has a nonnegative solution for $j=1,\ldots,r$ as desired. □

We know that the system of linear equations $A\mathbf{z}=\mathbf{b}$ has a solution if and only if $\mathbf{b}$ belongs to the image of A. For the later use, we give a more concrete description for the condition that $A\mathbf{z}=\mathbf{b}$ has a solution here.

Lemma 3.7. *Let $A=(a_{ij})\in\mathbb{R}^{m\times r}$, $\mathbf{b}=(b_j)\in\mathbb{R}^r$, and $\mathbf{z}=(z_1,\ldots,z_r)^T$, $\mathbf{y}=(y_1,\ldots,y_r)^T$ be sets of variables. Let $\{\mathbf{a}_1,\ldots,\mathbf{a}_m\}$ be the set of row vectors of A and let $I=(\mathbf{a}_1\mathbf{z}-y_1,\ldots,\mathbf{a}_m\mathbf{z}-y_m)\cap\mathbb{R}[y_1,\ldots,y_m]$. Assume that $\{\mathbf{c}_1\mathbf{y},\ldots,\mathbf{c}_l\mathbf{y}\}$ is a set of generators of I and let C be the matrix whose row vectors are $\{\mathbf{c}_1,\ldots,\mathbf{c}_l\}$. Then* $\mathrm{rank}(C)=m-\mathrm{rank}(A)$ *and $A\mathbf{z}=\mathbf{b}$ has a solution if and only if $C\mathbf{b}=\mathbf{0}$.*

Proof. Observe that $\{\mathbf{c}_1,\ldots,\mathbf{c}_l\}$ is a basis of the linear space of linear relationships among $\{\mathbf{a}_1,\ldots,\mathbf{a}_m\}$. In other words, $\{\mathbf{c}_1^T,\ldots,\mathbf{c}_l^T\}$ is a basis of the kernel of A^T. Thus $\mathrm{rank}(C)=\mathrm{rank}(\ker(A^T))=m-\mathrm{rank}(A)$.

If $C\mathbf{b}=\mathbf{0}$, i.e., $\mathbf{b}$ is a zero of the elimination ideal I, then by the Extension Theorem in p.125 of [1], we can extend $\mathbf{b}$ to a zero of the ideal $(\mathbf{a}_1\mathbf{z}-y_1,\ldots,\mathbf{a}_m\mathbf{z}-y_m)$. So $A\mathbf{z}=\mathbf{b}$ has a solution. The converse is obvious. □

Theorem 3.8. *Let $f_d=\sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i}-d\mathbf{x}^{\boldsymbol{\beta}}\in\mathbb{R}[\mathbf{x}]$, $\boldsymbol{\alpha}_i\in(2\mathbb{N})^n, c_i>0, i=1,\ldots,m$, $\boldsymbol{\beta}\in\mathrm{New}(f_d)^\circ\cap\mathbb{N}^n$, $\dim(\mathrm{New}(f_d))=n$, and d^* be defined as (3.1). Then $f_{d^*}\in$* SONC.

Proof. By Theorem 3.3, f_{d^*} has exactly one solution in $\mathbb{R}_+^n$, which is denoted by $\mathbf{x}_*$. Let

$$\{\Delta_1,\ldots,\Delta_r\}:=\{\Delta\mid\Delta\text{ is a simplex },\boldsymbol{\beta}\in\Delta^\circ,V(\Delta)\subseteq\{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_m\}\}$$

and $I_k:=\{i\in[m]\mid\boldsymbol{\alpha}_i\in V(\Delta_k)\}$ for $k=1,\ldots,r$. Firstly, we assume $\dim(\Delta_k)=n$ for all k. Hence $|I_k|=n+1$ for all k. For each Δ_k, since $\boldsymbol{\beta}\in\Delta_k^\circ$, we have $\boldsymbol{\beta}=\sum_{i\in I_k}\lambda_{ik}\boldsymbol{\alpha}_i$, where $\sum_{i\in I_k}\lambda_{ik}=1,\lambda_{ik}>0,i\in I_k$. Let us consider the following system of linear equations on variables c_{ik} and s_k:

$$\begin{cases}\frac{c_{ik}\mathbf{x}_*^{\boldsymbol{\alpha}_i}}{\lambda_{ik}}=s_k, & \text{for } i\in I_k,k=1,\ldots,r\\ \sum_{i\in I_k}c_{ik}=c_i, & \text{for } i=1,\ldots,m\end{cases}. \tag{3.10}$$

Eliminate c_{ik} from (3.10) and we obtain:

$$\sum_{i\in I_k}\lambda_{ik}s_k=c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i},\quad\text{for } i=1,\ldots,m. \tag{3.11}$$

Claim: The linear system (3.11) on variables $\{s_1,\ldots,s_r\}$ has a nonnegative solution.

Denote the coefficient matrix of (3.11) by A. Add up all of the equations of (3.11) and we obtain:

$$\sum_{k=1}^r s_k=\sum_{i=1}^m\sum_{i\in I_k}\lambda_{ik}s_k=\sum_{i=1}^m c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}. \tag{3.12}$$

Multiply the i-th equation of (3.11) by $\boldsymbol{\alpha}_i$ and then add up all of them. We obtain:

$$\boldsymbol{\beta}\sum_{k=1}^r s_k=\sum_{i=1}^m\sum_{i\in I_k}\lambda_{ik}\boldsymbol{\alpha}_i s_k=\sum_{i=1}^m c_i\boldsymbol{\alpha}_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}. \tag{3.13}$$

(3.13) $-$ (3.12) $\times\,\boldsymbol{\beta}$ gives

$$\sum_{i=1}^{m} c_i(\boldsymbol{\alpha}_i-\boldsymbol{\beta})\mathbf{x}_*^{\boldsymbol{\alpha}_i}=\mathbf{0}. \tag{3.14}$$

By Theorem 3.3, $\{c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}\}_{i=1}^m$ satisfies (3.14). Thus by Lemma 3.7, (3.11) has a solution. Moreover, since $\boldsymbol{\beta}\in\mathrm{New}(f_d)^\circ$ and $\dim(\mathrm{New}(f_d))=n$, then $\mathrm{rank}(\{\boldsymbol{\alpha}_i-\boldsymbol{\beta}\}_{i=1}^m)=n$. So $\mathrm{rank}(A)=m-n$.

For each j, denote the coefficient matrix of

$$\{\sum_{i\in I_k}\lambda_{ik}s_k=c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}\mid i\notin I_j\} \tag{3.15}$$

by A_j. For every $i\notin I_j$, since $\boldsymbol{\beta}\in\Delta_j^\circ$, there exists a facet F of Δ_j such that $\boldsymbol{\beta}\in\mathrm{conv}(V(F)\cup\{\boldsymbol{\alpha}_i\})^\circ$. Assume $\mathrm{conv}(V(F)\cup\{\boldsymbol{\alpha}_i\})=\Delta_{p_i}$. For every $k\notin\{j\}\cup\bigcup_{i\notin I_j}\{p_i\}$, let $s_k=0$ in (3.15) and we obtain:

$$\{\lambda_{ip_i}s_{p_i}=c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}\mid i\notin I_j\}. \tag{3.16}$$

Thus $\mathrm{rank}(A_j)=m-|I_j|=m-(n+1)=\mathrm{rank}(A)-1$. Therefore by Lemma 3.6, to prove the claim, we only need to show that the linear system (3.15) on variables $\{s_1,\dots,s_r\}\backslash\{s_j\}$ has a nonnegative solution for $j=1,\dots,r$.

Given $j\in[r]$, from (3.16) we have $s_{p_i}=c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}/\lambda_{ip_i}$ for $i\notin I_j$ and hence

$$\begin{cases} s_k=0, & \text{for } k\notin\{j\}\cup\bigcup_{i\notin I_j}\{p_i\}\\ s_{p_i}=c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}/\lambda_{ip_i}, & \text{for } i\notin I_j\end{cases} \tag{3.17}$$

is a nonnegative solution for (3.15). So the claim is proved.

Assume that $\{s_1^*,\dots,s_r^*\}$ is a nonnegative solution for the system of equations (3.11). Substitute $\{s_1^*,\dots,s_r^*\}$ into the system of equations (3.10), and we have $c_{ik}=\lambda_{ik}s_k/\mathbf{x}_*^{\boldsymbol{\alpha}_i}$ for $i\in I_k,k=1,\dots,r$. Let $d_k=s_k^*/\mathbf{x}_*^{\boldsymbol{\beta}}$ and $f_k=\sum_{i\in I_k}c_{ik}\mathbf{x}^{\boldsymbol{\alpha}_i}-d_k\mathbf{x}^{\boldsymbol{\beta}}$ for $k=1,\dots,r$. Then by (3.10) and by Proposition 2.4, d_k is the circuit number of f_k and f_k is a nonnegative circuit polynomial for all k. By (3.10), $\sum_{k=1}^r d_k\mathbf{x}_*^{\boldsymbol{\beta}}=\sum_{k=1}^r\sum_{i\in I_k}c_{ik}\mathbf{x}_*^{\boldsymbol{\alpha}_i}=\sum_{i=1}^m c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}=d^*\mathbf{x}_*^{\boldsymbol{\beta}}$. So we have $\sum_{k=1}^r d_k=d^*$. It follows $f_{d^*}=\sum_{k=1}^r f_k$, which proves the theorem.

For the case that $\dim(\Delta_k)=n$ does not hold for all k, note that all results above remain valid for $\boldsymbol{\beta}\in\mathbb{R}^n$. We give $\boldsymbol{\beta}$ a perturbation, say $\boldsymbol{\delta}$, such that $\dim(\Delta_k)=n$ holds for all k. Then the new linear system (3.11) for $\boldsymbol{\beta}+\boldsymbol{\delta}$ has a nonnegative solution. Let $\boldsymbol{\delta}\to\mathbf{0}$, we obtain that (3.11) also has a nonnegative solution for $\boldsymbol{\beta}$. Thus the theorem remains true in this case. $\square$

Theorem 3.9. *Let* $f_d=\sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i}-d\mathbf{x}^{\boldsymbol{\beta}}\in\mathbb{R}[\mathbf{x}]$, $\boldsymbol{\alpha}_i\in(2\mathbb{N})^n,c_i>0,i=1,\dots,m$, $\boldsymbol{\beta}\in\mathrm{New}(f_d)^\circ\cap\mathbb{N}^n$, $\dim(\mathrm{New}(f_d))=n$. *Then* f_d *is nonnegative if and only if* $f_d\in\mathrm{SONC}$.

Proof. The sufficiency is obvious. Assume that f_d is nonnegative. If $\boldsymbol{\beta}\in(2\mathbb{N})^n$ and $d<0$, or $d=0$, then f_d is a sum of monomial squares and obviously $f_d\in\mathrm{SONC}$. If $\boldsymbol{\beta}\notin(2\mathbb{N})^n$ and $d<0$, through a variable transformation $x_j\mapsto -x_j$ for some odd number β_j, we can always assume $d>0$. Let d^* be defined as (3.1). By Theorem 3.8, $f_{d^*}\in\mathrm{SONC}$. Suppose $f_{d^*}=\sum_{k=1}^r(\sum_{i\in I_k}c_{ik}\mathbf{x}^{\boldsymbol{\alpha}_i}-d_k\mathbf{x}^{\boldsymbol{\beta}})$, where $\sum_{i\in I_k}c_{ik}\mathbf{x}^{\boldsymbol{\alpha}_i}-d_k\mathbf{x}^{\boldsymbol{\beta}}$ is a circuit polynomials with d_k its circuit number for all k. Since f_d is nonnegative, then $d\le d^*$ by Theorem 3.3. We have

$f_d = \sum_{k=1}^{r}(\sum_{i\in I_k} c_{ik}\mathbf{x}^{\boldsymbol{\alpha}_i} - \frac{d}{d^*}d_k\mathbf{x}^{\boldsymbol{\beta}})$, where $\sum_{i\in I_k} c_{ik}\mathbf{x}^{\boldsymbol{\alpha}_i} - \frac{d}{d^*}d_k\mathbf{x}^{\boldsymbol{\beta}}$ is a nonnegative circuit polynomial for all k by Theorem 2.3. Thus $f_d \in \mathrm{SONC}$. $\square$

4. Nonnegative Polynomials with Multiple Negative Terms

In this section, we generalize Theorem 2.5 to general Newton polytopes.

Theorem 4.1. *Let $f = \sum_{i=1}^{m} c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^{l} d_j\mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$, $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 1,\dots,m$, $\boldsymbol{\beta}_j \in \mathrm{New}(f)^\circ \cap \mathbb{N}^n, j = 1,\dots,l$. Assume that all of the $\boldsymbol{\beta}_j$'s lie in the same side of all hyperplanes through any n points of $\{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_m\}$ and there exists a point $\boldsymbol{v} = (v_j) \in (\mathbb{R}^*)^n$ such that $d_j\boldsymbol{v}^{\boldsymbol{\beta}_j} > 0$ for all j. Then f is nonnegative if and only if $f \in \mathrm{SONC}$.*

Proof. Without loss of generality, assume $\dim(\mathrm{New}(f)) = n$ and $m > n+1$. The sufficiency is obvious. Suppose f is nonnegative. After a variable transformation $x_j \mapsto -x_j$ for all j with $v_j < 0$, we can assume all $d_j > 0$. Let

$$d_l^* \triangleq \sup\{d_l \in \mathbb{R} \mid f = \sum_{i=1}^{m} c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^{l-1} d_j\mathbf{x}^{\boldsymbol{\beta}_j} - d_l\mathbf{x}^{\boldsymbol{\beta}_l} \text{ is nonnegative}\}. \tag{4.1}$$

Note that d_l^* is well-defined since the set in (4.1) is nonempty and has upper bounds. Let $f^* = \sum_{i=1}^{m} c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^{l-1} d_j\mathbf{x}^{\boldsymbol{\beta}_j} - d_l^*\mathbf{x}^{\boldsymbol{\beta}_l}$. Then f^* is nonnegative and has a zero. Since for all $\mathbf{x} \in \mathbb{R}^n$, $f^*(|\mathbf{x}|) = \sum_{i=1}^{m} c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^{l-1} d_j|\mathbf{x}^{\boldsymbol{\beta}_j}| - d_l^*|\mathbf{x}^{\boldsymbol{\beta}_l}| \le f^*(\mathbf{x})$, $f^* = 0$ has a zero in $\mathbb{R}_+^n$, which is denoted by $\mathbf{x}_*$. Let

$$\{\Delta_1,\dots,\Delta_r\} := \{\Delta \mid \Delta \text{ is a simplex }, \boldsymbol{\beta}_j \in \Delta^\circ, j \in [l], V(\Delta) \subseteq \{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_m\}\}$$

and $I_k := \{i \in [m] \mid \boldsymbol{\alpha}_i \in V(\Delta_k)\}$ for $k = 1,\dots,r$. Because of the assumption of the theorem, $\dim(\Delta_k) = n$ for all k. For every $\boldsymbol{\beta}_j$ and every Δ_k, since $\boldsymbol{\beta}_j \in \Delta_k^\circ$, we have $\boldsymbol{\beta}_j = \sum_{i\in I_k} \lambda_{ijk}\boldsymbol{\alpha}_i$, where $\sum_{i\in I_k} \lambda_{ijk} = 1, \lambda_{ijk} > 0, i \in I_k$. Let us consider the following system of linear equations on variables c_{ijk}, d_{jk} and s_{jk}:

$$\begin{cases} \frac{c_{ijk}\mathbf{x}_*^{\boldsymbol{\alpha}_i}}{\lambda_{ijk}} = d_{jk}\mathbf{x}_*^{\boldsymbol{\beta}_j} = s_{jk}, & \text{for } i \in I_k, k = 1,\dots,r, j = 1,\dots,l \\ \sum_{k=1}^{r} d_{jk} = d_j, & \text{for } j = 1,\dots,l-1 \\ \sum_{k=1}^{r} d_{lk} = d_l^*, & \\ \sum_{j=1}^{l}\sum_{i\in I_k} c_{ijk} = c_i, & \text{for } i = 1,\dots,m \end{cases}. \tag{4.2}$$

Eliminate c_{ijk} and d_{jk} from (4.2) and we obtain:

$$\begin{cases} \sum_{j=1}^{l}\sum_{i\in I_k} \lambda_{ijk}s_{jk} = c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}, & \text{for } i = 1,\dots,m \\ \sum_{k=1}^{r} s_{jk} = d_j\mathbf{x}_*^{\boldsymbol{\beta}_j}, & \text{for } j = 1,\dots,l-1 \\ \sum_{k=1}^{r} s_{lk} = d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l}, & \end{cases}. \tag{4.3}$$

Claim: The linear system (4.3) on variables s_{jk} has a nonnegative solution.

Denote the coefficient matrix of (4.3) by A. Add up all of the equations of the first part of (4.3), and we obtain:

$$\sum_{j=1}^{l}\sum_{k=1}^{r} s_{jk} = \sum_{i=1}^{m}\sum_{j=1}^{l}\sum_{i\in I_k} \lambda_{ijk}s_{jk} = \sum_{i=1}^{m} c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}. \tag{4.4}$$

Multiply the i-th equation of the first part of (4.3) by $\boldsymbol{\alpha}_i$ and then add up all of them. We obtain:

$$\sum_{j=1}^{l}\boldsymbol{\beta}_j\sum_{k=1}^{r}s_{jk}=\sum_{i=1}^{m}\sum_{j=1}^{l}\sum_{i\in I_k}\lambda_{ijk}\boldsymbol{\alpha}_i s_{jk}=\sum_{i=1}^{m}c_i\boldsymbol{\alpha}_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}. \tag{4.5}$$

Substitute the second and the third part of (4.3) into (4.4) and (4.5), and we obtain:

$$\begin{cases}\sum_{j=1}^{l-1}d_j\mathbf{x}_*^{\boldsymbol{\beta}_j}+d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l}=\sum_{i=1}^{m}c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}\\ \sum_{j=1}^{l-1}d_j\boldsymbol{\beta}_j\mathbf{x}_*^{\boldsymbol{\beta}_j}+d_l^*\boldsymbol{\beta}_l\mathbf{x}_*^{\boldsymbol{\beta}_l}=\sum_{i=1}^{m}c_i\boldsymbol{\alpha}_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}\end{cases}. \tag{4.6}$$

The zero $\mathbf{x}_*$ of f^* is also the minimum of f^*. So it satisfies $f^*(\mathbf{x}_*)=\nabla(f^*(\mathbf{x}_*))=0$ which is (4.6). Thus by Lemma 3.7, (4.3) has a solution. Moreover, since $\dim(\Delta_1)=n$, the volume of Δ_1, which equals $\frac{1}{n!}|\det(\{\begin{pmatrix}1\\ \boldsymbol{\alpha}_i\end{pmatrix}\}_{i\in I_1})|$, is nonzero. So $\operatorname{rank}(\{\begin{pmatrix}1\\ \boldsymbol{\alpha}_1\end{pmatrix},\ldots,\begin{pmatrix}1\\ \boldsymbol{\alpha}_m\end{pmatrix},\begin{pmatrix}-1\\ -\boldsymbol{\beta}_1\end{pmatrix},\ldots,\begin{pmatrix}-1\\ -\boldsymbol{\beta}_l\end{pmatrix}\})=n+1$ and hence by Lemma 3.7, $\operatorname{rank}(A)=m+l-(n+1)>1$.

For every $u\in[l]$ and every $v\in[r]$, denote the coefficient matrix of

$$\begin{cases}\sum_{i\in I_k}\lambda_{ijk}s_{jk}=c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}, & \text{for } i\notin I_v\\ \sum_{k=1}^{r}s_{jk}=d_j\mathbf{x}_*^{\boldsymbol{\beta}_j}, & \text{for } j\neq u \text{ and } j\neq l\\ \sum_{k=1}^{r}s_{lk}=d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l}, & \end{cases} \tag{4.7}$$

by A_{uv}. For every $i\notin I_v$, since $\boldsymbol{\beta}_u\in\Delta_v^\circ$, there exists a facet F of Δ_v such that $\boldsymbol{\beta}_u\in\operatorname{conv}(V(F)\cup\{\boldsymbol{\alpha}_i\})^\circ$. Assume $\operatorname{conv}(V(F)\cup\{\boldsymbol{\alpha}_i\})=\Delta_{p_i}$. For $j=u,k\notin\cup_{i\notin I_v}\{p_i\}$ or $j\neq u,k\neq v$, let $s_{jk}=0$ in (4.7), and we obtain:

$$\begin{cases}\lambda_{iup_i}s_{up_i}=c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}, & \text{for } i\notin I_v\\ s_{jv}=d_j\mathbf{x}_*^{\boldsymbol{\beta}_j}, & \text{for } j\neq u \text{ and } j\neq l\ .\\ s_{lv}=d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l}, & \end{cases} \tag{4.8}$$

Thus $\operatorname{rank}(A_{uv})=m-|I_v|+l-1=m-(n+1)+l-1=\operatorname{rank}(A)-1$. Therefore by Lemma 3.6, to prove the claim, we only need to show that the linear system (4.7) on variables $\{s_{jk}\}_{j,k}\backslash\{s_{uv}\}$ has a nonnegative solution for all $u\in[l]$ and all $v\in[r]$.

Given $v\in[r]$, from (4.8) we have $s_{up_i}=c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}/\lambda_{iup_i}$ for $i\notin I_v$ and $s_{jv}=d_j\mathbf{x}_*^{\boldsymbol{\beta}_j}$ for $j\neq u$. Hence

$$\begin{cases}s_{jk}=0, & \text{for } j=u,k\notin\cup_{i\notin I_v}\{p_i\} \text{ or } j\neq u,k\neq v\\ s_{up_i}=c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}/\lambda_{iup_i}, & \text{for } i\notin I_v\\ s_{jv}=d_j\mathbf{x}_*^{\boldsymbol{\beta}_j}, & \text{for } j\neq u \text{ and } j\neq l\\ s_{lv}=d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l}, & \end{cases} \tag{4.9}$$

is a nonnegative solution for (4.7). So the claim is proved.

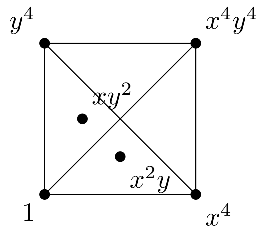
Assume that $\{s_{jk}^*\}_{j,k}$ is a nonnegative solution for the system of equations (4.3). Substitute $\{s_{jk}^*\}_{j,k}$ into the system of equations (4.2), and we have $c_{ijk}=\lambda_{ijk}s_{jk}/\mathbf{x}_*^{\boldsymbol{\alpha}_i}$ for $i\in I_k,k=1,\ldots,r,j=1,\ldots,l$. Let $f_{jk}=\sum_{i\in I_k}c_{ijk}\mathbf{x}^{\boldsymbol{\alpha}_i}-d_{jk}\mathbf{x}^{\boldsymbol{\beta}_j}$ for $k=1,\ldots,r,j=1,\ldots,l$. Then by (4.2) and by Proposition 2.4, d_{jk} is the circuit number of f_{jk} and f_{jk} is a nonnegative circuit polynomial for all j,k. By (4.2), we have $f=\sum_{j=1}^{l-1}\sum_{k=1}^{r}f_{jk}+\sum_{k=1}^{r}(\sum_{i\in I_k}c_{ilk}\mathbf{x}^{\boldsymbol{\alpha}_i}-\frac{d_l}{d_l^*}d_{lk}\mathbf{x}^{\boldsymbol{\beta}_l})$. Since $d_l\leq d_l^*$,

$\sum_{i\in I_k} c_{ilk}\mathbf{x}^{\boldsymbol{\alpha}_i} - \frac{d_l}{d_l^*} d_{lk}\mathbf{x}^{\boldsymbol{\beta}_l}$ is a nonnegative circuit polynomial for all k by Theorem 2.3. Thus $f \in \mathrm{SONC}$. $\square$

The condition that all of the $\boldsymbol{\beta}_j$'s lie in the same side of all hyperplanes through any n points of $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}$ in Theorem 4.1 is necessary. We give an example to illustrate this.

Example 4.2. *Let* $d^* = \sup\{d \in \mathbb{R}_+ \mid 50x^4y^4 + x^4 + 3y^4 + 800 - 300xy^2 - dx^2y$ *is nonnegative*$\}$ *and* $f = 50x^4y^4 + x^4 + 3y^4 + 800 - 300xy^2 - d^*x^2y$*. Then* $f \notin \mathrm{SONC}$*.*

Proof. $f = \nabla(f) = 0$ has exactly one zero $(x_* = 4.03172, y_* = 0.424042, d = 184.828)$ in $\mathbb{R}_+^n$. $(1,2) = \frac{1}{4}(4,4) + \frac{1}{4}(0,4) + \frac{1}{2}(0,0) = \frac{1}{4}(4,0) + \frac{1}{2}(0,4) + \frac{1}{4}(0,0)$, $(2,1) = \frac{1}{4}(4,4) + \frac{1}{4}(4,0) + \frac{1}{2}(0,0) = \frac{1}{2}(4,0) + \frac{1}{4}(0,4) + \frac{1}{4}(0,0)$.



By the proof of Theorem 4.1, if $f \in \mathrm{SONC}$, then the following linear system

$$\begin{cases} 50x_*^4y_*^4 = \frac{1}{4}s_1 + \frac{1}{4}s_3 \\ x_*^4 = \frac{1}{4}s_2 + \frac{1}{4}s_3 + \frac{1}{2}s_4 \\ 3y_*^4 = \frac{1}{4}s_1 + \frac{1}{2}s_2 + \frac{1}{4}s_4 \\ 800 = \frac{1}{2}s_1 + \frac{1}{4}s_2 + \frac{1}{2}s_3 + \frac{1}{4}s_4 \\ 300x_*y_*^2 = s_1 + s_2 \\ dx_*^2y_* = s_3 + s_4 \end{cases} \tag{4.10}$$

on variables $\{s_1, s_2, s_3, s_4\}$ should have a nonnegative solution. However, (4.10) has no nonnegative solutions. So $f \notin \mathrm{SONC}$. $\square$

The condition that there exists a point $\boldsymbol{v} \in (\mathbb{R}^*)^n$ such that $d_j\boldsymbol{v}^{\boldsymbol{\beta}_j} < 0$ for all j in Theorem 4.1 is necessary. We give an example to illustrate this.

Example 4.3. *Let* $d^* = \sup\{d \in \mathbb{R}_+ \mid x^6y^6 + x^6 + y^6 + 1 - x^2y^3 - xy^2 + dxy^3$ *is nonnegative*$\}$ *and* $f = x^6y^6 + x^6 + y^6 + 1 - x^2y^3 - xy^2 + d^*xy^3$*. Then* $f \notin \mathrm{SONC}$*.*

Proof. Suppose $f \in \mathrm{SONC}$ and let $f = \sum_{i=1}^6 f_k$, where $f_1 = a_1x^6y^6 + c_1y^6 + d_1 - e_1x^2y^3, f_2 = b_1x^6 + c_2y^6 + d_2 - e_2x^2y^3, f_3 = a_2x^6y^6 + c_3y^6 + d_3 - g_1xy^2, f_4 = b_2x^6 + c_4y^6 + d_4 - g_2xy^2, f_5 = a_3x^6y^6 + c_5y^6 + d_5 + h_1xy^3, f_6 = b_3x^6 + c_6y^6 + d_6 + h_2xy^3$ are nonnegative circuit polynomials. Assume that (x^*, y^*) is a zero of f. We have $f_k(x^*, y^*) = 0$ for $k = 1, \ldots, 6$. By $f_1(x^*, y^*) = f_2(x^*, y^*) = 0$, we have $y^* > 0$. By $f_3(x^*, y^*) = f_4(x^*, y^*) = 0$, we have $x^* > 0$. By $f_5(x^*, y^*) = f_6(x^*, y^*) = 0$, we have $x^*y^* > 0$. It is a contradictory. Hence $f \notin \mathrm{SONC}$. $\square$

Corollary 4.4. *Let* $f = \sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^l d_j\mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$*,* $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 1, \ldots, m$*,* $\boldsymbol{\beta}_j \in \mathrm{New}(f)^\circ \cap \mathbb{N}^n, d_j > 0, j = 1, \ldots, l$*,* $\dim(\mathrm{New}(f)) = n$*. Assume that* f *is nonnegative and has a zero, and all of the* $\boldsymbol{\beta}_j$*'s lie in the same side of all* *hyperplanes through any n points of $\{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m\}$. Then f has exactly one zero in $\mathbb{R}^n_+$.*

Proof. By Theorem 4.1, $f \in$ SONC. Suppose $f = \sum_{k=1}^r f_k$, where all f_k are nonnegative circuit polynomial. Let $\mathbf{x}$ be a zero of f. Then $f_k(\mathbf{x}) = 0$ for all k. By Proposition 2.4, $f_k(|\mathbf{x}|) = 0$ and hence $f(|\mathbf{x}|) = 0$. Again by Proposition 2.4, f_k has exactly one zero in $\mathbb{R}^n_+$ for all k. So f has exactly one zero in $\mathbb{R}^n_+$. □

5. Computation via Relative Entropy Program

In this section, we put the problem to decide $f \in$ SONC down to the feasibility of a relative entropy program (REP). Since REPs are convex, they can be checked very efficiently.

Theorem 5.1. *(*[3, Theorem 3.2]*) Let $f = \sum_{i=1}^m c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}}$ be a circuit polynomial, which is not a sum of monomial squares. Assume $\boldsymbol{\beta} = \sum_{i=1}^m \lambda_i \mathbf{x}^{\boldsymbol{\alpha}_i}$, where $\sum_{i=1}^m \lambda_i = 1, \lambda_i > 0, i = 1, \dots, m$. Then f is nonnegative if and only if the following REP on variables ν_i and δ_i is feasible:*

$$\begin{cases} \textit{minimize} & 1 \\ \nu_i = d\lambda_i, & \textit{for } i = 1, \dots, m \\ \nu_i \log(\nu_i / c_i) \le \delta_i, & \textit{for } i = 1, \dots, m \\ \sum_{i=1}^m \delta_i \le 0, & \end{cases} \tag{5.1}$$

We make the following assumption for the rest of this section.

Assumption: Let $f = \sum_{i=1}^m c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^l d_j \mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$, $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 1, \dots, m$, $\boldsymbol{\beta}_j \in \mathrm{New}(f)^\circ \cap \mathbb{N}^n, d_j > 0, j = 1, \dots, l$. For every $\boldsymbol{\beta}_j$, let

$$\{\Delta_{j1}, \dots, \Delta_{js_j}\} := \{\Delta \mid \Delta \text{ is a simplex }, \boldsymbol{\beta}_j \in \Delta^\circ, V(\Delta) \subseteq \{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m\}\}$$

and $I_{jk} := \{i \in [m] \mid \boldsymbol{\alpha}_i \in V(\Delta_{jk})\}$ for $k = 1, \dots, s_j$ and $j = 1, \dots, l$. For every $\boldsymbol{\beta}_j$ and every Δ_{jk}, since $\boldsymbol{\beta}_j \in \Delta^\circ_{jk}$, we can write $\boldsymbol{\beta}_j = \sum_{i \in I_{jk}} \lambda_{ijk} \mathbf{x}^{\boldsymbol{\alpha}_i}$, where $\sum_{i \in I_{jk}} \lambda_{ijk} = 1, \lambda_{ijk} > 0, i \in I_{jk}$.

Theorem 5.2. *Let $f = \sum_{i=1}^m c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^l d_j \mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$, $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 1, \dots, m$, $\boldsymbol{\beta}_j \in \mathrm{New}(f)^\circ \cap \mathbb{N}^n, d_j > 0, j = 1, \dots, l$. Then $f \in$ SONC if and only if the following REP on variables d_{jk}, ν_{ijk}, c_{ijk} and δ_{ijk} is feasible:*

$$\begin{cases} \textit{minimize} & 1 \\ \nu_{ijk} = d_{jk}\lambda_{ijk}, & \textit{for } i \in I_{jk}, k = 1, \dots, s_j, j = 1, \dots, l \\ \nu_{ijk} \log(\nu_{ijk}/c_{ijk}) \le \delta_{ijk}, & \textit{for } i \in I_{jk}, k = 1, \dots, s_j, j = 1, \dots, l \\ \sum_{i \in I_{jk}} \delta_{ijk} \le 0, & \textit{for } k = 1 \dots, s_j, j = 1, \dots, l \\ \sum_{j=1}^l \sum_{i \in I_{jk}} c_{ijk} = c_i, & \textit{for } i = 1, \dots, m \\ \sum_{k=1}^{s_j} d_{jk} = d_j, & \textit{for } j = 1, \dots, l \end{cases} \tag{5.2}$$

Proof. Suppose that $f_{jk} = \sum_{i \in I_{jk}} c_{ijk} \mathbf{x}^{\boldsymbol{\alpha}_i} - d_{jk} \mathbf{x}^{\boldsymbol{\beta}_j}$ is a nonnegative circuit polynomial for $k = 1, \dots, s_j, j = 1, \dots, l$ and $f = \sum_{j=1}^l \sum_{k=1}^{s_j} f_{jk}$. Then by Theorem 5.1, $(d_{jk})_{j,k}$, $(\nu_{ijk})_{i,j,k} = (d_{jk}\lambda_{ijk})_{i,j,k}$, $(c_{ijk})_{i,j,k}$ and $(\delta_{ijk})_{i,j,k} = (\nu_{ijk} \log(\nu_{ijk}/c_{ijk}))_{i,j,k}$ is a feasible solution of (5.2).

Conversely, suppose that $(d_{jk})_{j,k}$, $(\nu_{ijk})_{i,j,k}$, $(c_{ijk})_{i,j,k}$ and $(\delta_{ijk})_{i,j,k}$ is a feasible solution of (5.2). Let $f_{jk} = \sum_{i \in I_{jk}} c_{ijk} \mathbf{x}^{\boldsymbol{\alpha}_i} - d_{jk} \mathbf{x}^{\boldsymbol{\beta}_j}$ for $k = 1, \dots, s_j, j = 1, \dots, l$.

Then by Theorem 5.1, f_{jk} is a nonnegative circuit polynomial for all j, k. Moreover, we have $f = \sum_{j=1}^{l} \sum_{k=1}^{s_j} f_{jk}$. Thus, $f \in$ SONC. $\square$

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